

Critical currents and self organization in quantum Hall bilayers

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We present a theory of the critical interlayer tunneling current in a disordered quantum Hall bilayer at total filling factor one, allowing for the effect of static vortices. In agreement with recent experiments [Phys. Rev. B **80**, 165120 (2009); *ibid.* **78**, 075302 (2008)], we find that this critical current is proportional to the sample area and is comparable in magnitude to observed values. This reflects the formation of a Bean critical state as a result of current injection at the boundary. We predict a crossover to a critical current proportional to the square-root of the area in smaller samples. We also predict a peak in the critical current as the electron density varies at fixed layer separation.

In a quantum Hall bilayer at total Landau level filling $\nu_T = 1$, Coulomb interactions induce a state with interlayer phase coherence [1, 2], which can be understood as a Bose-Einstein condensate of interlayer excitons [3, 4]. The motion of excitons corresponds to counterflowing electrical currents in the layers, so that excitonic supercurrents can give dissipationless electrical transport. In particular, a finite interlayer current I at negligible interlayer voltage V has been predicted [5, 6] and recently confirmed by four-terminal measurements [7, 8]. This behavior, which is a form of Josephson effect [9], also appears as a zero-bias peak in differential conductivity measurements [10, 11]. The low-voltage regime terminates when the interlayer current exceeds a critical value I_c . Dissipation increases dramatically above I_c .

A key question has emerged over the nature of the critical current I_c . Recent experiments have shown that it is proportional to the sample area [8, 12]. Within the simplest model of a clean homogeneous bilayer, this scaling can only be explained by assuming that the tunnel splitting is several orders of magnitude smaller than expected [13]. For realistic splittings, tunneling is estimated to occur within a few microns of the contact [Eq. (4)] so that I_c should not depend on the sample length in the direction of the current [13–15]. Edge tunneling [16] has also been proposed, predicting linear scaling with the sample length but not its area.

In this paper, we present a theory which produces a critical current [Eq. (7)] that is proportional to the area of the sample and of the correct order of magnitude, given reasonable estimates for the parameters. A disorder-induced lengthscale, L_d , emerges in our theory [Eq. (5)]; this scale has no counterpart in the clean system [13]. Our results are also consistent with the observed dependence of I_c on the magnetic length and on an in-plane magnetic field. A key test of our theory is the prediction that I_c should scale with the square-root of the sample size for samples smaller than L_d .

The essential feature of our work is that we allow for static vortices in the exciton superfluid, which will be nucleated by strong charge disorder [17–20]. Such vortices

play a crucial role for the critical current. They pin any injected supercurrents and sustain dissipationless states, in much the same way that disorder pins magnetic flux in superconductors [21–23], or charge in charge-density waves [24]. However, there is a significant difference in the bilayer: the depinning force comes from the injected charge current. Since these currents cannot penetrate the bulk of the quantum Hall state, the depinning force is applied *only* at the sample boundary. Given this geometry, it is a surprising feature of our results that the critical current I_c can scale with the sample area.

We will first present numerical results showing that, in one dimension, currents injected at the boundary decay linearly in space in the disordered state (Fig. 1). This is analogous to the Bean critical state in a superconductor. We argue that these special critical states are generated by current injection from the boundary, and hence are selected in the bilayer. Based on this numerical work, we will then present a heuristic argument for the critical current, which we generalize to two dimensions and compare with the experiments.

Our starting point is the energy functional

$$H_{\text{eff}} = \int \left[\frac{\rho_s}{2} (\nabla \phi)^2 - t \cos(\phi + \theta^0) \right] d^D \mathbf{r}, \quad (1)$$

for the low-energy modes of a bilayer containing pinned vortices. This form follows from the clean model [5] when the phase field of the pinned vortices θ^0 is subtracted out of the superfluid phase θ : $\phi = \theta - \theta^0$. The first term in Eq. (1) is the superfluid stiffness while the second describes the interlayer tunneling. We assume that the vortex field θ^0 is disordered, with a correlation length ξ which we estimate to be of the order of the distance to the dopants, $d_d \approx 200$ nm in current samples [17]. The counterflow supercurrent density above the ground state, j_{CF} , and the interlayer tunneling current density, j_t , are related to the phase field by:

$$j_{\text{CF}} = \frac{e\rho_s}{\hbar} \nabla \phi, \quad j_t = et \sin(\phi + \theta^0). \quad (2)$$

A time-varying superfluid phase $\phi(t)$ gives rise to an interlayer voltage difference V *via* the Josephson relation

$V = \hbar \dot{\phi}/e$. Therefore, a state with a finite interlayer current at zero interlayer voltage is time-independent, corresponding to a local minimum of the energy (1). To investigate this possibility, we consider the dissipative model

$$-\lambda \dot{\phi} = \frac{\delta H_{\text{eff}}}{\delta \phi} = -\rho_s \nabla^2 \phi + t \sin(\phi + \theta^0), \quad (3)$$

whose stationary solutions $\dot{\phi} = V = 0$ are the local minima of Eq. (1). The stationary equation is the continuity equation stating that the loss of counterflow current (first term) is accounted for by interlayer tunneling (second term). The dissipative dynamics of Eq. (3) is physically a resistive shunt due to interlayer quasiparticle tunneling. Additional dynamical terms are necessary in the finite voltage regime [14, 19, 20, 25]. However, it is the dissipation that determines the long-time limit at low voltages, so that here we may use Eq. (3).

The boundary conditions for Eq. (1) come from the current flows through the sample [13]. For definiteness, we consider a tunneling geometry in which a current I_t is injected into the top layer at one corner and removed from the bottom layer at the opposite corner. These current flows may be written as superpositions of layer-symmetric and layer-antisymmetric currents,

$$I_{\text{in(out)}} = \frac{I_t}{2} [(1, 1) \pm (1, -1)]$$

where the components refer to the two layers. Thus, the tunneling experiment corresponds to a flow of layer-symmetric current, with equal counterflow currents $I = I_t/2$ injected by both the electron source and drain. In the low-voltage regime, these counterflow currents will be the supercurrents j_{CF} in Eq. (2). Since the symmetric component cannot penetrate the bulk, the supercurrents are injected at the boundary. As we shall see, the profile of their injection along the boundary is unimportant.

The mechanisms leading to an extensive critical current for the bilayer can be seen most clearly from a one-dimensional model. In Fig. 1, we show current and voltage profiles obtained for the one-dimensional version of Eq. (3), on a lattice of $L = 200$ sites. These results are averaged over realizations of the disorder θ_i^0 , which is taken independently on different lattice sites. This corresponds to using the correlation length ξ of the continuum model as the lattice spacing. The natural unit of current is then $I_0 = e\rho_s/\hbar\xi$. For this illustration, we take the tunneling strength $t\xi^2/\rho_s = 0.4$. In each realization, we start from an initial state in which the ϕ_i are random and independent, and equilibrate by integrating forwards in time with the boundary conditions $\partial_x \phi|_1 = \partial_x \phi|_L = 0$. To model the current injection in a tunneling experiment, we then slowly increase the boundary conditions to the final values $\xi \partial_x \phi|_1 = -\xi \partial_x \phi|_L = I/I_0$. For the lowest three values of I used, the dynamics reach a time-independent solution, corresponding to the Josephson

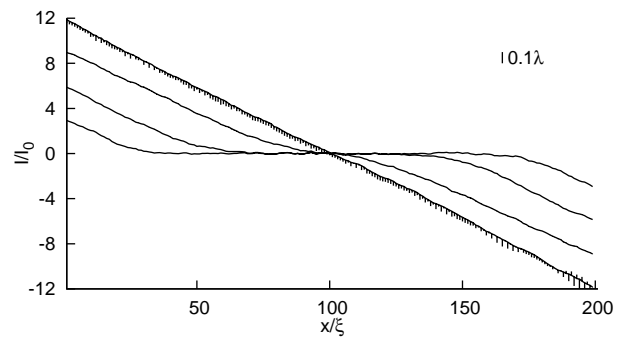


FIG. 1: Profile of counterflow current I (lines) and interlayer voltage ($V \propto \dot{\phi}$, vertical bars, non-zero in top curve only) from (3) on a one-dimensional lattice with current injection at both ends, at a time $\approx 10^4/\lambda$ after the current is switched on. The injected counterflow currents at each boundary are 3, 6, 9, $12I_0$ for the four curves ($I_0 = e\rho_s/\hbar\xi$). The interlayer voltages vanish below a critical current I_c . Here, $9 < I_c/I_0 < 12$. $t\xi^2/\rho_s = 0.4$, results averaged over 50 realizations.

regime with vanishing interlayer voltages. For too large I , these time-independent solutions break down and the phase winds continuously at late times. This corresponds to the breakdown of the d.c. Josephson regime and the appearance of a state with finite interlayer voltages.

The static states in Fig. 1 differ qualitatively from those of the clean model [14, 26], $\theta^0 = 0$. In that case, there is a penetration depth for the injected current of

$$\lambda_J \sim \sqrt{\rho_s/t}. \quad (4)$$

Since the phase angle is periodic this implies a maximum injected current density of $\partial_x \phi \sim \pi/\lambda_J$. In the disordered case shown, however, the injected current decays linearly close to the contacts, with a slope which is independent of the injected current. Thus, an increase in the injected current is accommodated by an increased current penetration into the sample. As can be seen from the figure, this process continues until the currents fill the entire sample. Beyond this point, further increases in current cannot be accommodated by coherent tunneling and an interlayer voltage develops. Since the current decays linearly with a constant slope, the resulting I_c in the one-dimensional model scales with the sample length.

In the clean model, the breakdown of the stationary solutions can be understood in terms of the injection of phase solitons at the boundary [14, 27], which propagate through the sample. Thus, the phase at any point varies in time, and the system develops an interlayer voltage by the a.c. Josephson effect. However, for the disordered system, the solitons may be pinned by disorder. We now develop a heuristic theory of such pinning, which agrees with our numerical work.

We begin by recalling [24, 28] the form of the ground states of the random field XY model, Eq. (1), in the weak disorder regime $\xi \ll \lambda_J$ relevant for the bilayer. In this

regime, the ground state consists of ferromagnetic domains with polarized phase. The key idea is that it is energetically costly to have phase twists at scales shorter than the size, L_d , of these domains. The energy cost for a phase twist that varies over the scale L_d is $\rho_s L_d^{D-2}$ in D dimensions. The typical tunneling energy of a polarized domain is obtained by summing random energies in the range $\pm t\xi^D$ for its $(L_d/\xi)^D$ correlation areas, giving $t\xi^D(L_d/\xi)^{D/2}$. Thus, for $d < 4$, the phase stiffness wins at short scales, and the domains reach a finite size where the two energies balance. This gives the Imry-Ma scale:

$$L_d \sim \left(\frac{\rho_s}{t\xi^{D/2}} \right)^{\frac{2}{4-D}} = \left(\frac{\lambda_J^2}{\xi^{D/2}} \right)^{\frac{2}{4-D}}. \quad (5)$$

In this ground state of polarized domains, the average coarse-grained phase over a domain is chosen such that the tunneling energy H_t of each domain is minimized. Since $\delta H_t / \delta \phi(\mathbf{r})$ is the tunneling current at position \mathbf{r} , the total tunneling current over the domain vanishes.

We now consider how the system changes in the presence of an injected current. The injected counterflow will cause the phase to twist away from its equilibrium value, leading to finite tunneling currents. We assume that the configuration remains smooth on the scale L_d , and hence average Eq. (3) over each domain. The second term in Eq. (3) becomes $(t/L_d^D)(L_d/\xi)^{D/2}f(\bar{\phi})$ where $\bar{\phi}$ is the deviation of the coarse-grained phase from its equilibrium value, and the range of $f(\bar{\phi})$ is typically $[-1, 1]$. For a dissipationless state ($\dot{\phi} = 0$), the coarse-grained phase $\bar{\phi}$ should therefore obey:

$$-L_d^2 \nabla^2 \bar{\phi} + f(\bar{\phi}) = 0. \quad (6)$$

The source term in Eq. (6) describes the loss of injected current due to tunneling in a domain. As discussed above, current injection induces counterflow currents and hence phase twists. Since it is energetically costly to introduce phase twists in a domain, the domain at the boundary will respond by rotating uniformly, increasing its tunneling current, thereby reducing the counterflow current. This process continues until the tunneling in the domain saturates, so that $|f| \sim 1$. The residual counterflow currents will be transmitted further into the sample, causing the domains there to rotate in a similar way. Thus, we argue that forcing at a boundary leads to a self-organized critical state, in which the driven part of the system sits at the threshold $|f| \sim 1$.

In one dimension, this argument means that Eq. (6) would give an average counterflow current $\rho_s \nabla \bar{\phi}$ that decreases linearly from the boundary. To be more precise, it predicts a linear I/I_0 in the saturated regions, with a slope $-\xi^{-1}(t\xi^2/\rho_s)^{4/3}$. This is qualitatively consistent with the numerical results shown in Fig. 1. Note that, in one dimension, Eq. (6) describes a harmonic chain with random static friction [29]. The process described above

is simply the transmission of forces when such a chain is pushed at its ends.

As we now describe, the generalization of this argument to two dimensions will account for the critical current of the bilayer, as measured by Tiemann *et al.* [8]. (We see similar behavior in two-dimensional simulations, albeit with large disorder fluctuations.) In our scenario of saturated domains, the static state only breaks down when the final domain in the sample exceeds threshold. Therefore, we may determine the critical current by setting $f = 1$ everywhere in Eq. (6). Integrating over space, we see that the critical current, defined as the total injected current at threshold, functionally counts the number of domains in the sample, and is independent of the precise geometry. We find, for its order of magnitude,

$$I_c \sim \frac{e\rho_s}{\hbar} \frac{S}{L_d^D}, \quad (7)$$

where S is the sample area in 2D and sample length in 1D. Note that this is a natural form for an extensive critical current, composed of the system size, the characteristic length L_d , and the microscopic current scale $e\rho_s/\hbar$. The same form can be seen in Eq. (13) of Ref. 20. However, it is not clear whether the result there applies to a bilayer driven at its boundary.

For Eq. (7) to apply, the sample should comprise many domains. If a dimension L_x is smaller than the domain size L_d then the second term in Eq. (6) should be multiplied by $\sqrt{L_d/L_x}$, because the total tunneling current of the domain is cut off at the sample width L_x . This gives

$$I_c \sim \frac{e\rho_s}{\hbar} \sqrt{\frac{L_x}{L_d}} \frac{L_y}{L_d} \quad (\text{quasi-1D: } L_x \ll L_d \ll L_y). \quad (8)$$

Similarly, for a sample containing only a single domain,

$$I_c \sim \frac{e\rho_s}{\hbar} \sqrt{\frac{L_x L_y}{L_d^2}} \quad (\text{for } L_x, L_y \ll L_d). \quad (9)$$

To compare Eq. (7) with the experiments, we start from the microscopic theory [30] for the zero-temperature values of the stiffness, ρ_s^0 , and order parameter, $m_x = \langle \cos \phi \rangle$, for a homogeneous bilayer. The latter renormalizes the tunneling strength t^0 so that $t^0 = \Delta_0 m_x / 2\pi l^2$ where Δ_0 is the single-particle tunnel splitting. This theory does not predict accurately the critical layer separation for the loss of interlayer coherence but should be reasonable away from the critical point. Owing to the strong charge disorder, we expect the incompressible quantum Hall phase to occupy a small fraction of the sample, with the remainder occupied by puddles of compressible electron liquid [17, 18]. We suppose that the incompressible phase forms a network [18] of channels of size l separating puddles of size $\xi \approx d_d$, the distance to the dopants. Thus, the effective parameters in the energy [Eq. (1)] for the disordered bilayer should be $t \sim (l/d_d)t^0$

and $\rho_s \sim (l/d_d)\rho_s^0$ ($\approx 20\text{mK}$ at a layer separation $d \approx l$). The microscopic current scale is then $e\rho_s/\hbar \approx 0.5\text{ nA}$.

Figure 2 shows our estimates for the domain size L_d and the critical current I_c , as functions of the ratio of interlayer separation to magnetic length, d/l . We see that the domains ($L_d^2 \lesssim 0.01\text{ mm}^2$) are indeed not larger than the samples areas of $(0.01 - 1)\text{ mm}^2$ so that the results are consistent with area scaling for I_c . Moreover, I_c has the correct order of magnitude compared to the observed values [8] of $0.1 - 10\text{ nA}$.

Interestingly, I_c has a peak as a function of d/l which is also suggested in the experimental data [8]. This feature appears robust: it arises from the increase in L_d as d/l is reduced, caused by the increase in ρ_s . However, the peak position depends on the precise variation of model parameters with d/l . For example, the variation in ρ_s may cause some variation in vortex density [17] and hence ξ , pushing the peak to smaller values of d/l .

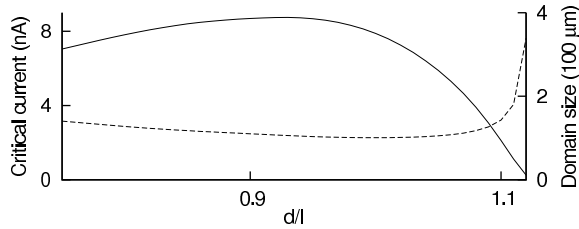


FIG. 2: Predicted critical current (solid, left axis) and domain size (dotted, right axis), for area $S = 0.1\text{ mm}^2$, $d = 28\text{ nm}$, ξ ($\approx d_d$) = 200 nm , and bare tunneling $\Delta_0 = 150\text{ }\mu\text{K}$. The stiffness and tunneling renormalization are taken from theory [30], and scaled by the network geometrical factor (see text).

Our theory also has implications for the effect of an in-plane magnetic field B_{\parallel} . This introduces a lengthscale $l_{\parallel} = hc/eB_{\parallel}d$, which is the length of a loop enclosing a flux quantum in the cross-section of a bilayer. Within our theory, the system should be insensitive to B_{\parallel} unless $l_{\parallel} < \xi$. At such fields, circulating tunneling currents are set up between the two layers, so that the *net* tunneling current in a domain is reduced. Therefore, coherent tunneling should be suppressed for $B_{\parallel} > hc/ed\xi \sim 0.7\text{ T}$, which is consistent with experiments [31] where the enhanced tunneling decreases above 0.5 T .

In conclusion, we have presented a theory of the critical interlayer currents in a disordered quantum Hall bilayer with static pinned vortices. We find that, because the current is injected at the boundary, coherent tunneling saturates in the current-carrying region, leading to a Bean critical state. This results in an extensive critical current for sufficiently large samples (in contrast to the clean limit [13] where area scaling holds for small samples). The magnitude of the critical current is consistent with experiments. We predict that area scaling does not hold when the samples become smaller than the phase-pinned domains, and also that the critical current peaks in the interlayer coherent phase.

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